Realistic Mathematical Modeling and Problem Posing

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ABSTRACT: In this paper I will present a teaching experiment characterized by a sequence of activities based on the use of suitable cultural artifacts, interactive teaching methods, and the introduction of new socio-mathematical norms in order to create a substantially modified teaching/learning environment. The focus is on fostering i) a mindful approach toward realistic mathematical modelling, and ii) a problem posing attitude. I will argue for modelling as a means of recognizing the potential of mathematics as a critical tool to interpret and understand the communities children live, or society in general. Teaching students to interpret critically the reality they live in and to understand its codes and messages so as not to be excluded or misled, should be an important goal for compulsory education.

Introduction

In previous studies (see e.g. Bonotto, 1995) I have analyzed some difficulties regarding the understanding of the structure of decimal numbers, in particular the conceptual obstacles elementary and middle school students encounter in mastering the meaning of decimal numbers and in ordering sequences of decimal numbers. The findings of some investigations administered to children in grade 5, 6 and 8, are consistent with those of classical researches (Nesher & Peled, 1986; Resnick et al., 1989) and more recent studies (e.g., Stacey & Steinle, 1999; Irwin, 2001).

The results of two questionnaires, involving elementary and middle school Italian teachers and concerning the way they teach the topic of decimal numbers in class, shed light on the way the usual instructional practice seems totally extraneous to the richness of the experiences students develop outside school (Bonotto, 1996). Many teachers introduce decimal numbers by extending the place-value convention. They tend to spend little time allowing children to understand the meaning of decimal numeration or reflect on decimal number properties and relationships. As a consequence, children learn to carry out the required computations, but have difficulty in mastering the relationship between symbols and their referents, and between fractional and decimal representations.

In agreement with other researchers (e.g. Hiebert, 1985; Irwin, 2001) I believe that the decimal numbers usually taught in school need to be anchored to students' existing knowledge in some way. Teachers should be able to help students integrate their everyday knowledge with school mathematics. Thus it is important that teachers recognize the numerical culture acquired outside the scholastic environment in order to offer children the opportunity to develop new mathematical knowledge preserving the focus on meaning found in everyday situations.

The study presented in this paper involves a teaching experiment based on a sequence of classroom activities in upper elementary school aimed at enhancing the understanding of the structure of decimal numbers in a way that was meaningful and consistent with a disposition towards making sense of numbers (Sowder, 1992). As in our other studies [see e.g. Bonotto 2003, 2005 and 2007a] the classroom activities are based on an extensive use of suitable cultural artifacts, in this case some menu of restaurants and pizzerias. The classroom activities are also based on the use of a variety of complementary, integrated, and interactive teaching methods, and on the introduction of new socio-mathematical norms, in the sense e.g. of Yackel & Cobb

(1996), in an attempt to create a substantially modified teaching/learning environment. This environment is focused on fostering a mindful approach toward realistic mathematical modeling and a problem posing attitude.

Theoretical and empirical background

The habit of connecting mathematics classroom activities with everyday-life experience is still substantially delegated to word problems. However, besides representing the interplay between in- and out-of-school contexts, word problems are often the only example providing students with a basic sense experience in mathematization and mathematical modeling. Several empirical researches have documented that the practice of word problem solving in school mathematics promotes in students the exclusion of realistic considerations and a "suspension" of sense-making, and rarely reaches the idea of mathematical modeling (see Verschaffel, Greer, & De Corte, 2000, for a review of these studies).

"*Rather than functioning as realistic contexts that invite or even force pupils to use their common-sense knowledge and experience about the real world, school arithmetic word problems have become artificial, puzzle-like tasks that are perceived as being separate from the real world. Thus, pupils learn that relying on common-sense knowledge and making realistic considerations about the problem context—as one typically does in real-life problem situations encountered outside school—is harmful rather than helpful in arriving at the 'correct' answer of a typical school word problem"*, Verschaffel, De Corte, and Borghart (1997, p.339).

Several studies point to two reasons for this lack of use of everyday-life knowledge: textual factors relating to the stereotypical nature of the most frequently used textbook problems

"*When problem solving is routinised in stereotypical patterns, it will in many cases be easier for the student to solve the problem than to understand the solution and why it fits the problem*", Wyndhamn and Säljö, 1997, p.364,

and presentational or contextual factors associated with practices, environments and expectations related to the classroom culture of mathematical problem solving

"*In general the classroom climate is one that endorses separation between school mathematics and every-day life reality*", Gravemeijer, 1997, p.389.

Finally, in my opinion, another reason for the abstention from using realistic considerations is that the practice of word problem solving is relegated to classroom activities, having meaning and location, in terms of time and space, only within the school; rarely will students encounter these activities in this form outside of school (Bonotto, 2007a).

I deem that an early introduction in schools of fundamental ideas about modelling is not only possible but also indeed desirable even at the primary school level. In this contribution the term mathematical modeling is not only used to refer to a process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original (real-world) situation, evaluated and communicated. Besides this type of modeling, which requires that the student has already at his disposal at least some mathematical models and tools to mathematize, there is another kind of modeling, wherein model-eliciting activities are used as a vehicle for *the development* (rather than the application) of mathematical concepts. This second type of modeling is called 'emergent modeling' (Gravemeijer, 2007). Although it is very difficult, if not impossible, to make a sharp distinction between the two aspects of mathematical modeling, it is clear that they are associated with different phases in the teaching/learning process and with different kinds of instructional activities (Greer, Verschaffel & Mukhopadhyay, 2007). However, in this contribution the focus will be more addressed to the second aspect of mathematical modeling.

To implement an early introduction in elementary schools of fundamental ideas about realistic mathematical modeling, and for laying the foundations of a mathematization disposition, I deem that we have to create more realistic and less stereotyped problem situations. These should be more closely related to children's experiential world and meaningful. I deem that an extensive use of suitable cultural artifacts, with their incorporated mathematics, can play a fundamental role in bringing students' out-of-school reasoning experiences into play, by creating a new tension between school mathematics and everyday-life knowledge (Bonotto, 2007b). [1]The cultural artifacts we introduced into classroom activities (see for example Bonotto 2003, 2005 and 2007a) are concrete materials, real or reproduced, which children typically meet in real-life situations. In this way we can offer the opportunity of making connections between the mathematics incorporated in real-life situations and school mathematics. These artifacts are relevant and meaningful to children because they are part of their real life experience, offering significant references to out-of-school experiences. In this way we can enable children to keep their reasoning processes meaningful and to monitor their inferences. Finally, I believe that certain cultural artifacts lend themselves naturally to helping students with problem posing activities.

Problem posing is an important aspect of both pure and applied mathematics and an integral part of modelling cycles which require the mathematical idealization of real world phenomenon (Christou et al. 2005). For this reason, problem posing is of central importance in the discipline of mathematics and in the nature of mathematical thinking and it is an important companion to problem solving. Recently many mathematics educators realized that developing the ability to pose mathematics problems is at least as important, educationally, as developing the ability to solve them and have underlined the need to incorporate problem-posing activities into mathematics classrooms; also documents promoting curricular and pedagogical innovation in mathematics education have recently called for an increased emphasis on problem posing activities in mathematics classroom (e.g. NCTM, 2000), i.e. of activities in which students generate their own problems in addition to solving pre-formulated problems. Given the importance of problem-posing activities in school mathematics, some researchers started to investigate various aspects of problem-posing processes (Silver, 1994; Silver et al., 1996; Ellerton & Clarkson, 1996; English, 1998 and 2003; Christou et al. 2005). Several studies have reported approaches to incorporate problem posing in instruction. Some studies provided evidence that problem posing has a positive influence on students' ability to solve word problems and provided a chance to gain insight into students' understanding of mathematical concepts and processes. It was found that students' experience with problem posing enhances their perception of the subject, provides good opportunities for children to link their own interests with all aspects of their mathematics education, and can prepare students' to be intelligent users of mathematics in their everyday lives.

[1] Cole (1995) points out that an essential property of artifacts, which supports their bilateral influence and offers common bases to culture and discourse, is that they contain in coded form the interactions of which they were previously a part and which they mediate in the present.

Problem posing is seen here as a classroom activity which is important both from the cognitive and the metacognitive viewpoint. Children's expression of mathematical ideas through the creation of their own mathematics problems demonstrates not only their understanding and level of concept development, but also their perception of the nature of mathematics (Ellerton et al., 1996) and their attitude towards this discipline.

Problem posing has been defined by researchers from different perspectives (see e.g. Silver et al., 1996). In this paper I consider mathematical problem posing as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. This process is similar to situations to be mathematized, which students have encountered or will encounter outside school. According to English (1998) "*we need to broaden the types of problem experiences we present to children … and, in so doing, help children "connect" with school mathematics by encouraging everyday problem posing (Resnick et a., 1991). We can capitalize on the informal activities situated in children's daily lives and get children in the habit of recognizing mathematical situations wherever they might be*".

THE STUDY

The Basic Characteristics of the Teaching/Learning Environment

The basic characteristics of the teaching/learning environment designed and implemented in the classroom, and that were also present in other studies we have conducted [see e.g. Bonotto 2003, 2005], are as follows:

1. The creation of more realistic and less stereotyped problem situations based on the use of suitable cultural artifacts, for example labels or supermarket receipts, that can constitute a didactic interface between in and out-of-school experience and knowledge, by favoring students' out-of-school reasoning experiences to come into play.
2. The application of a variety of complementary, integrated and interactive instructional techniques (involving children's own written descriptions of the methods they use, individual or in pairs working, whole class discussions, and the drafting of a text by the whole class).
3. An attempt to establish a new classroom culture also through new socio-mathematical norms, for example the norms about what counts as a good or acceptable response or solution procedure in order to undermine some deeply rooted and counterproductive beliefs and attitudes such as mathematics problems have only one right answer or there is only one correct way to solve any mathematical problem.

For an analysis of these three aspects, in particular of the role of the cultural artifact as mathematizing tool that preserve the focus on meaning found in everyday situations see for example Bonotto, 2005 and 2007b.

Participants/ Materials/Procedure

The study was carried out in two fourth-grade classes (children 9-10 years of age) in a lakeside resort in the north of Italy by the official logic-mathematics teacher, in the presence of a research-teacher. As a control, two fourth-grade classes were chosen in the same town.

Most of the local population are involved in tourism and catering, and the children's parents were for the most part either owned or worked in restaurants, bars, ice-cream parlors or pizza shops; for this reason it was found that the pricelists and menus of pizza shops, fast food and normal restaurants were part of the children's experiential reality.

The teaching experiment was subdivided into six sessions, each lasting two hours, at weekly intervals. The first session was devoted to the administration of the pre-test and the introduction of various kinds of artifacts which the children, divided into groups, had to analyze, by reading and interpreting all the data present, whether numerical or not. Sessions 2 to 6 concerned five experiences involving different opportunities offered by the artifacts. The sixth session was also devoted to administration of the post-test. Sessions 2 to 6 were divided into two phases. In the first, each pupil was given an assignment to carry out individually or in pairs. In the second phase, the results obtained were discussed collectively and the various answers and strategies compared.

Data

The research method was both qualitative and quantitative. The qualitative data consisted of students' written work, fields notes of classroom observations and mini-interviews with students after the experiences. Quantitative date was collected by means of pre- and post-tests, administered to the two experimental classes as well as the other two control classes. The two tests were constructed by taking some items normally used in the bimonthly tests utilized by the same teachers or usually present in the textbook.

Research questions and hypotheses

The first general hypothesis was that the teaching experiment class fostered the understanding of some aspects of the multiplicative structure of decimal numbers in a way that was meaningful and consistent with a disposition towards making sense of numbers (Sowder, 1992). The hypothesis is that this understanding was greater in the experimental classes than in the control classes, who received more traditional teaching. I believe that this would be due to the opportunity the children had to refer to a concrete reality (via the cultural artifact), explore their strategies and compare them with those of their schoolmates, the use of estimation and approximation processes, as well as thanks to an adequate balance between problem posing and problem solving activities and requirements.

Furthermore, I hypothesized that, contrary to the common practice of word-problem solving, children in this teaching experiment would not ignore the relevant, plausible and familiar aspects of reality, nor would they exclude real-world knowledge from their observations and reasoning (hypothesis II). Then children would also exhibit flexibility in their reasoning, by exploring different strategies, often sensitive to the context and quantities involved, in a way that was meaningful and closer to the procedures emerging from out-of-school mathematics practice (hypothesis III).

Finally I wanted to evaluate the impact of problem posing activities, and to analyze if and how a particular use of suitable cultural artifacts, with its incorporated mathematics, can play a role to favoring also problem posing activities and not only problem solving activities as already studied in previous our researches [Bonotto e Basso 2001; Bonotto 2003, 2005 and 2007a].

Some results

The experience of such a study proved a fruitful one not only from the cognitive viewpoint [and the quantitative results of the post-test confirmed our first hypothesis] but also from the metacognitive one.

By presenting the students with activities that are meaningful because they involve the use of material familiar to them increased their motivation to learn even among the less able ones. A good example is the case of an immigrant child with learning difficulties related chiefly to linguistic problems. For her, as for many others, being confronted with a well-known everyday object with "*few words and lots of numbers*" acted as a stimulus. Indeed, it led her to say *"It's easier than the problems in the book because we already know how things work at a restaurant!"* This confirms what was stated in Bonotto (2005) "*Roughly speaking using a receipt, which is poor in words but rich in implicit meanings, overturns the usual buying and selling problem situation, which is often rich in words but poor in meaningful references*".

From certain class discussions there emerged some interesting observations on the understanding of how decimal numbers and the euro are written. But, significantly, a process of *problem critiquing* (English, 1998) was set up, whereby the children attempted to solve, criticize and make suggestions or correct the problems created by their classmates. Here is an example taken from the second session. In this session, the children read and interpreted the data and information in the various menus (products on offer, prices, ingredients, cover charges etc.). Working individually on one of these menus, the children compiled a hypothetical order, which they themselves chose according to their experience outside school (see Figure 1). In so doing, they had to follow the structural features of a blank receipt (description of goods, quantity, cost etc.) provided by the teacher. Finally, the children had to calculate how much they would have to pay, by adding up the bill.

Figure 1

DESCRIZIONE	QUANTITA'
FRUTTI DI MARE	2
SCALOPPINE AL LIM.	1
INSALATA MISTA	1
FRUTTA FRESCA	1
ACQUA NATURALE	1
COCA COLA MEDIA	1

I. Look at this. Do you think it could be a kind of problem?
P 737. It's not written…
I. You're right… It's not written, but all the data is there. We could write the text … Shall we try?
P 734. A man goes to a restaurant and orders 2 dishes of sea-food, 1plate of escallops in lemon sauce, 1 mixed salad, 1 fresh fruit, 1 still mineral water, 1 medium Coke. How much does he spend?
I. Does everyone agree?
P 740. You can't do it like that because there are no prices … We have to put a menu underneath the problem or put the prices in the text.
P 725. And then… how do we know how many people were eating if they didn't put the cover charge?

The idea of "*pretending to be at a restaurant"* and *"acting like grown-ups" ["grown-ups don't take a calculator or a pencil and paper with them to see if they can afford to order this or that …. They work it out in their heads*" said one child, P725] helped the children, including those with greater difficulties, to reason more freely and adopt calculation strategies they had never used before. Furthermore by placing more emphasis on the processes of understanding by means of dialogue and discussion, and less on whether the answer was right or wrong, the negative impact of a wrong answer was reduced. The atmosphere within the class was one of cooperation and healthy competition which made pupils do their best. "*It didn't even feel like math!!!*" was the ecstatic exclamation of some children conveying their enthusiasm for the experience.

Let us look at an example taken from the fifth session. In the fifth session, the children were given a more complex menu (including, for example, various "supplements"). They were asked: i) to analyze the menu and to read and interpret all the data and information contained therein; ii) to choose what to order knowing that they had only 15 euros to spend; iii) to make a mental estimate of what they would have to pay and whether it would come within their budget; iv) to write out the bill in full to check their estimate.

Here is an example, P734, of a child considered "low level" who had been placed in the "extra help group".

"First of all, I take away the money for the cover charge, which is obligatory. So € 15.00 – € 1.25 is like doing € 15.00 – € 1.00 which makes 14.00 but it's a bit less because it was € 1.25... so we can pretend that I've still got € 13.50 for example. Then I decided to have the cheapest pizza, the pizza marinara, so I have to take away € 3.35. A quick way is to take away € 3.50 and so I'd still have about € 10.00 and I can add something on the pizza, for example ham which costs € 1.44. Let's pretend it's 1.50, so € 10.00 - € 1.00 makes € 9.00 and then taking away another € 0.50 it makes € 8.50. In the end, I can have a Coke as well which costs € 3.00 and that makes € 5.50... (a brief pause*)... But I wanted a dessert and now I can't afford it... I have to leave something out... (he thinks for a few seconds)... I need another euro because the desserts cost € 6.50.... Perhaps it's better to have water instead of a Coke. That way instead of € 3.00 I only spend € 2.00 for drinks and now I've got € 6.50 which is the price of a 'semifreddo' dessert"*.

As can be seen, P 734 is unable to calculate mentally € 15.00 - € 1.25 but by approximating upwards all the prices, he is able to find a solution to the problem after his first failed attempt.
Like P 734, all the other children, without exception, managed to carry out this exercise. Most of them preferred to use subtraction and calculate the amount they had left in order to decide if they should order something else and what to order. Only two children preferred to use addition and calculate as they went along the amount of the final bill. Here is an example:

"I started with the cover charge which is € 1.25 then I added the pizza siciliana which is € 5.94... but to do it quickly I pretended it was €6.00 and so the total was € 7.25. Now I can get something to drink. I decided to have water which costs less so that I could perhaps manage to have a dessert afterwards. So I say € 7.25 + € 2.00 = € 9. 25. All the desserts cost € 6.50 each. If I do € 9.25 + 6.50= 9.00 + 6.00 = 15.00 but then I have to add on the 25 cents and the 50 cents and so it's to much because I've only got € 15.00 to spend. So, I think I'll have a cheaper pizza, for example a margherita which costs € 3.61 (say 3.50). So, I have to start again by doing 1.25 + 3.50= 3.00 + 1.00= 4.00 + 25 cents = 4.25 + 50 cents = 4.75 cents. Now I add on the water which is 4.75 + 2.00 = 6. 75 and try again adding the dessert doing 6.75 + 6.50= 6 + 6= 12.00 + 50 cents = 12. 50 + 75 cents which is about one euro so it's more or less € 13.50".

The less confident children (as in the case reported above) made an estimate that was less precise, whereas the more confident ones attempted a more careful estimate down to the number of cents. Here is another example, P 740:

"I first took away the price of the cover charge, so I did € 15.00 – 1.25 = 14.00. Then I took away another € 0.25 which is half of 0.50: € 15.00 – 0.50 = € 13.50 + 0.25= € 13.75. Then I decided to take away the price of a dessert which is what I like most, so I do € 13.75 - € 6.50. Here, I can do what I did before and divide 75 cents in two and do 50 and 25. So 13.50 – 6.50= 7.50 €. But I've got to add on the 25 cents and so € 7.50 + 25= € 7.75. Now I can have a Coke which is easy because I have to take away 3

euros and that makes € 4.75. With that money I can have a pizza Napoli which costs € 4.64 and still have some money left."

These examples clearly demonstrate that the children have also exhibited flexibility in their reasoning, by exploring different strategies, often sensitive to the context and quantities involved, in a way that was meaningful and consistent with a sense-making disposition and closer to the procedures emerging from out-of-school mathematics practice (hypothesis III confirmed). Many other examples of written works demonstrated that the children did ignore the relevant, plausible and familiar aspects of reality, nor they excluded real-world knowledge from their observations and reasoning (also hypothesis II confirmed).

Conclusion and open problems

In this paper I have presented a teaching experiment characterized by a sequence of activities based on the use of suitable cultural artifacts, interactive teaching methods, and the introduction of new socio-mathematical norms. An effort was made to create a substantially modified teaching/learning environment that focused on fostering a mindful approach towards realistic mathematical modeling and problem posing. I will argue for modelling as a means of recognizing the potential of mathematics as a critical tool to interpret and understand the communities children live, or society in general. Teaching students to interpret critically the reality they live in and to understand its codes and messages so as not to be excluded or misled, should be an important goal for compulsory education. The positive results of the teaching experiment can be attributed to a combination of closely linked factors, in particular the use of suitable cultural artifacts and an adequate balance between problem posing and problem solving activities.

Regarding the use of cultural artifacts the implementation of this kind of classroom activity requires a radical change on the part of teachers as well (for an analysis see Bonotto, 2005). These tools differ from those usually mastered by the teacher that are quite always highly structured, rigid, not really suitable to develop alternative processes deriving from circumstantial solicitations, unforeseen interests, particular classroom situations. The teacher has to be ready to *create* and manage open situations, that are continuously *transforming*, that can be mastered after long experimentation and of which he/she cannot foresee the final evolution or result. As a matter of fact, these situations are sensitive to the social interactions that are established, to the students reactions, their ability to ask questions, to find links between school and extra-school knowledge; hence the teacher has to be able to modify on the way the contents objectives of the lesson. In this way the class cannot be prepared in advance in all of its aspects, nor from above; it should rather plan for various "branches" to be then drawn together through a process whose management is quite hard.

In future research, we will take a further look at the role of cultural artifacts not only as mathematizing tools that keep the focus on meaning found in everyday situations and as tools of mediation and integration between in and out-of-school knowledge, but also as possible interface tools between problem posing and problem solving activities.

Regarding the problem posing activities I deem that a problem-posing program, which integrates the promising outcomes of recent studies (English, 1998; English, Cudmore & Tilley, 1998), should permeate the entire curriculum. According to English (2003)

Problem posing—like its companion, problem solving—should be an integral component of the mathematics curriculum across all content domains. Although problem posing occurs naturally in everyday life, it does not receive due attention in the mathematics classroom. Problem posing is more than just constructing problems. Children apply problem-posing processes when they are actively engaged in problematic situations that involve them in exploring, questioning, constructing, and refining mathematical ideas and relationships.

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